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MODULARITY OF GENERATING SERIES OF WINDING NUMBERS

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ABSTRACT. The Shimura correspondence connects modular forms of integral weights and half-integral weights. One of the directions is realized by the Shintani lift, where the inputs are holomorphic differentials and the outputs are holomorphic modular forms of half-integral weight. In this article, we generalize this lift to differentials of the third kind. As an application we obtain a modularity result concerning the generating series of winding numbers of closed geodesics on the modular curve.

1. INTRODUCTION

For an arithmetic subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$, the quotient $M = \Gamma \backslash \mathbb{H}$ is a (possibly non-compact) Riemann surface. Denote by M^* the standard compactification of M . On this real surface, there is an ample supply of closed geodesics associated to indefinite binary quadratic forms. Given any differential 1-form η on M^* , it is natural to pair this form with these geodesics, and study the generating series of these numbers. This was initiated by Shintani for holomorphic 1-forms $\eta = f(z)dz$ coming from a holomorphic cusp form f [21]. He viewed M^* as a locally symmetric space associated to the orthogonal group of signature $(2, 1)$, and considered the integral

$$(1.1) \quad I(\tau, \eta) := \int_{M^*} \eta \wedge \Theta(\tau, z, \varphi) d\bar{z},$$

where $\Theta(\tau, z, \varphi)d\bar{z}$ is a theta function valued in differential 1-forms. The output is the generating series of cycle integrals of f , and also a modular form of half-integral weight corresponding to f under the Shimura correspondence [20]. This construction has been extended to much greater generality by Kudla and Millson [17, 18] for closed, rapidly decaying differential forms η on locally symmetric spaces associated to orthogonal and unitary groups to produce holomorphic Siegel and Hermitian modular forms. These results can be considered as analogues of the classical result of Hirzebruch and Zagier [14].

In this paper, we study the case when η is a differential 1-form of the third kind on M^* . The integral $I(\tau, \eta)$ now needs to be suitably regularized (see (3.27)). The output will be a real-analytic modular form of weight $3/2$, whose holomorphic part consists of cycle integrals of η . In particular, if M^* has genus zero, then these cycle integrals are closely related to the

winding number of the closed geodesics around the poles of η . To see this, we take as an example $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and consider

$$(1.2) \quad \eta := \frac{j'(z)}{j(z) - 1728} dz,$$

where $j : M^* \rightarrow \mathbb{P}^1(\mathbb{C})$ is the Klein j -invariant.

For a positive discriminant D , let \mathcal{Q}_D be the set of integral, binary quadratic forms $[A, B, C] = Ax^2 + Bxy + Cy^2$ of discriminant D , which has an action by Γ . One can associate a closed geodesic c_X to each $X \in \mathcal{Q}_D$, whose image $c(X)$ in M^* is a closed curve depending only on the Γ -equivalent class of X . Let $\mathrm{Ind}(X)$ be the winding number of $j(c(X))$ with respect to 1728 and ∞ on $\mathbb{P}^1(\mathbb{C})$, where we count with half multiplicity if $j(c(X))$ passes through 1728 or ∞ . As a special case of our main theorem, we show that the generating functions of certain averages of these winding numbers possess modular properties.

Theorem 1.1. *Let $\Delta < 0$ be a fixed fundamental discriminant, χ_Δ the associated genus character (see (3.35)), and $L(s, \Delta)$ the Dirichlet L -function associated to the Dirichlet character $\left(\frac{\Delta}{\cdot}\right)$. Then the generating series*

$$(1.3) \quad G_\Delta(\tau) := L(0, \Delta) + \sum_{\substack{d>0 \\ d \equiv 0,3 \pmod{4}}} q^d \sum_{X \in \Gamma \backslash \mathcal{Q}_{-d\Delta}} \chi_\Delta(X) \mathrm{Ind}(X)$$

is a mixed mock modular form in the sense of [7] (see section 3.1) of weight $3/2$ on $\Gamma_0(4)$. Its shadow is the Siegel indefinite theta function

$$\theta_\Delta(\tau) := \frac{1}{\sqrt{|\Delta|}} \sum_{A,B,C \in \mathbb{Z}} \chi_\Delta([A, B, C]) (A + C) q^{-(B^2 - 4AC)/|\Delta|} e^{-4\pi v(B^2 + (A-C)^2)/|\Delta|}.$$

In general, we can consider the higher level analogue, where Γ and \mathcal{Q}_D are replaced by the congruence subgroup $\Gamma_0(N)$ and the subset

$$\mathcal{Q}_{N,D} := \{[A, B, C] \in \mathcal{Q}_D : N \mid A\} \subset \mathcal{Q}_D$$

respectively. In this case, the period integrals of an arbitrary differential of the third kind η (with residue divisor with integral coefficients) are often transcendental by results of Scholl, Waldschmidt, and Wüstholz, see e.g. [19, 24, 26]. Therefore the coefficients in the generating series above will in general not be algebraic. Nevertheless, the generating series can still be made modular by the following general result, of which Theorem 1.1 is a special case.

Theorem 1.2. *Let N be a positive integer and $\Delta < 0$ a fundamental discriminant satisfying $\Delta \equiv r^2 \pmod{4N}$, and χ_Δ the genus character as defined in (3.35). For an arbitrary differential of the third kind η with residue divisor $\sum_{\zeta \in M^*} r_\zeta(\eta) \cdot [\zeta]$ and $X \in \mathcal{Q}_{N,D}$, denote $\int_{c(X)} \eta$ the cycle integral of η along the geodesic on M^* determined by X (see (2.19)). We define*

the twisted trace of index d for η by

$$\mathrm{Tr}_{N,\Delta,d}(\eta) := \sum_{X \in \Gamma_0(N) \backslash \mathcal{Q}_{N,-d\Delta}} \frac{\chi_\Delta(X)}{2\pi i} \int_{c(X)} \eta.$$

Then the generating series

$$(1.4) \quad G_{\Delta,N}(\tau, \eta) := \sum_{\substack{d > 0 \\ d \equiv 0, 3 \pmod{4}}} \mathrm{Tr}_{N,\Delta,d}(\eta) q^d$$

becomes a real-analytic modular form of weight $3/2$ on $\Gamma_0(4N)$ after adding the function $\Theta_\Delta^*(\tau, \eta)$ in (3.39), whose image under the lowering operator is the indefinite Siegel theta function $\Theta_\Delta(\tau, \eta)$ in (3.40).

Remark 1.3. The function $\Theta_\Delta^*(\tau, \eta)$ is the evaluation of a period integral of the Siegel theta function at the residue divisor of η . When η is holomorphic, the residue divisor of η is trivial and this term vanishes identically. In this case, which was originally considered by Shintani [21], the generating series $G_{\Delta,N}(\tau, \eta)$ is already modular.

Remark 1.4. Theorem 1.1 follows from Theorem 1.2 by specializing to $N = 1$ and with residue divisor $[\infty] - [\sqrt{-1}]$ on the modular curve, which is the locally symmetric space associated to a specific even, integral lattice (see Example 2.1). Both results are special cases of Theorem 3.7, which also includes the case of a Shimura curve (see also Section 5).

Suppose that the residue divisor in Theorem 1.2 is given by $[\zeta_1] - [\zeta_2]$, the difference of two points in M^* . Then the theta lift $I(\tau, \eta)$ is closely related to the indefinite theta functions ϑ^{c_1, c_2} constructed by Zagier [29] and Kudla and the second author [9] associated to two (arbitrary) preimages c_1 and c_2 of $[\zeta_1]$ and $[\zeta_2]$ in the extended upper half plane \mathbb{H}^* under the quotient map $\mathbb{H}^* \rightarrow M^*$. Explicitly, ϑ^{c_1, c_2} can be realized by integrating the Kudla-Millson theta form over any curve in \mathbb{H}^* connecting c_2 with c_1 . Then the two non-holomorphic forms $I(\tau, \eta)$ and ϑ^{c_1, c_2} differ by a holomorphic cusp form, more precisely by a (cohomological) Shintani lift. In particular, $I(\tau, \eta)$ and ϑ^{c_1, c_2} coincide if the genus of M^* is zero.

The paper is organized as follows. First, we recall the basic setup in Section 2. Then we will prove the main result (Theorem 3.7), and deduce Theorem 1.2 from it. After that we relate the integral $I(\tau, \eta)$ to the indefinite theta function of Zagier and Funke-Kudla. Finally in the last section, we specialize Theorem 3.7 to the case of a Shimura curve and give an example related to Ramanujan's mock theta function.

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2. SETUP

2.1. Modular Curve. Let B be an indefinite quaternion algebra over \mathbb{Q} and let $V = B^0 \subset B$ be the subspace of elements of trace zero. For N a positive rational number, there is a quadratic form $Q(\alpha) := -N\text{Nm}(\alpha)$ on V , where Nm is the reduced norm on B . This turns V into a quadratic space of signature $(2, 1)$ and $G = \text{Spin}(V)$ is an algebraic group over \mathbb{Q} .

Since B is indefinite, the real points $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$ of V are isomorphic to

$$M_2(\mathbb{R})^0 := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\},$$

where the norm on B is identified with the determinant and the action of $\text{Spin}(V)(\mathbb{R})$ becomes that of $\text{SL}_2(\mathbb{R})$ via

$$g \cdot X := gXg^{-1},$$

for $X \in M_2(\mathbb{R})^0$. We can identify the symmetric space associated to V with the Grassmannian of oriented negative lines in $V_{\mathbb{R}}$, and we denote by \mathcal{D} its connected component containing the line spanned by $\frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then \mathcal{D} can be identified with the upper half-plane \mathbb{H} as follows. For $z = x + iy \in \mathbb{H}$, let $g_z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ be the element such that $g_z \cdot i = z$. Then $X(z) := \frac{1}{\sqrt{N}} g_z \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{N}y} \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}$ spans a negative line in $V_{\mathbb{R}}$. It is easy to check that $\gamma \cdot X(z) = X(\gamma z)$. Furthermore, we denote

$$(2.1) \quad W(z) := \partial_{\bar{z}} X(z) = \frac{1}{2i\sqrt{N}y^2} \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix};$$

$$(2.2) \quad R(X, z) := (X, X) + \frac{1}{2}(X, X(z))^2.$$

Let $(\cdot, \cdot)_z$ be the majorant associated to z ; a small calculation gives

$$(2.3) \quad (X, X)_z = (X, X) + (X, X(z))^2.$$

Note that the quantity $(X, X(z))$ never vanishes for nonzero X when $Q(X) \leq 0$.

For $L \subset V$ an even, integral lattice with quadratic form Q and dual lattice L' , let $\Gamma_L \subset \text{SO}(L)$ be the discriminant kernel and $\Gamma \subset \Gamma_L \cap \text{SO}^+(V_{\mathbb{R}})$ the subgroup in the connected component $\text{SO}^+(V_{\mathbb{R}})$ of $\text{SO}(V_{\mathbb{R}})$ containing identity. The quotient $M = M_L := \Gamma \backslash \mathcal{D}$ is a modular curve or Shimura curve. The latter occurs exactly when B does not split over \mathbb{Q} , that is, $\text{Iso}(V)$, the set of rational isotropic lines in V , is empty. If M is non-compact, then M can be compactified by adding $\Gamma \backslash \text{Iso}(V)$. In that case we denote $\mathcal{D}^* := \mathcal{D} \cup \text{Iso}(V)$ and $M^* = M_L^* := \Gamma \backslash \mathcal{D}^*$.

Example 2.1. Suppose that $V(\mathbb{Q}) = M_2(\mathbb{Q})^0$. Then $\text{SL}_2 \cong \text{Spin}(V)$ as algebraic groups over \mathbb{Q} . There is also a bijection between $\mathbb{P}^1(\mathbb{Q})$ and $\text{Iso}(V)$ by sending $(\alpha : \beta)$ to the line

spanned by $\begin{pmatrix} -\alpha\beta & \alpha^2 \\ -\beta^2 & \alpha\beta \end{pmatrix}$. This identifies \mathcal{D}^* with $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, where the action of $\mathrm{SO}(V)$ on \mathcal{D}^* becomes the linear fractional transformation on \mathbb{H}^* . Consider the lattice

$$L := \left\{ \begin{pmatrix} -B & -C/N \\ A & B \end{pmatrix} \in V : A, B, C \in \mathbb{Z} \right\} \subset V.$$

Its dual lattice is given by

$$L' := \left\{ \begin{pmatrix} -B/2N & -C/N \\ A & B/2N \end{pmatrix} \in V : A, B, C \in \mathbb{Z} \right\}.$$

Then $\Gamma = \Gamma_0(N)$ and $\mathrm{SO}^+(L)$ is the extension of $\Gamma_0(N)$ by all Atkin-Lehner involutions, see Section 2.4 of [3].

2.2. Cusps and Fundamental Domains. Suppose that V splits over \mathbb{Q} , i.e. $V(\mathbb{Q}) = M_2(\mathbb{Q})^0$. Let $\ell_0 \in \mathrm{Iso}(V)$ be the line spanned by $u_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which corresponds to the cusp $i\infty$. For any $\ell \in \mathrm{Iso}(V)$, let $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ be an element such that $\ell = \sigma_\ell \ell_0$. Set $u_\ell := \sigma_\ell u_0$. If $\Gamma_\ell \subset \Gamma$ is the stabilizer of ℓ , then

$$(2.4) \quad \sigma_\ell^{-1} \Gamma_\ell \sigma_\ell = \left\{ \pm \begin{pmatrix} 1 & k\alpha_\ell \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\}$$

with $\alpha_\ell \in \mathbb{Z}_{>0}$ the width of the cusp ℓ . Choose $\beta_\ell \in \mathbb{Q}_{>0}$ such that $\beta_\ell u_\ell$ is a primitive element of $\ell \cap L$, and write $\varepsilon_\ell := \alpha_\ell / \beta_\ell$. For $h \in L'/L$, we can also define $h_\ell \in \mathbb{Q}_{\geq 0}$ to be the quantity that satisfies $0 \leq h_\ell < \beta_\ell$ and $\ell \cap (L + h) = (\mathbb{Z}\beta_\ell + h_\ell)u_\ell$. We use $[\ell]$ to represent the class of ℓ in the finite set $\Gamma \backslash \mathrm{Iso}(V)$. Note that $\alpha_\ell, \beta_\ell, \varepsilon_\ell$ and h_ℓ are all class invariants.

For $\epsilon > 0$ and $z \in M$, let $B_\epsilon(z)$ be the open ϵ -neighborhood with respect to the hyperbolic metric. For $\ell \in \mathrm{Iso}(V)$, we also have an open neighborhood $B_\epsilon(\ell)$ near ℓ given by $q_\ell^{-1}(U_{\epsilon^{1/\alpha_\ell}})$, where $U_r := \{w \in \mathbb{C} : |w| < r\}$ and $q_\ell(z) = \mathbf{e}(\sigma_\ell^{-1}z/\alpha_\ell)$. These charts turns $M^* := \Gamma \backslash \mathcal{D}^*$ into a compact Riemann surface.

For a finite subset $S = S^\circ \cup S_\infty \subset M^*$ with $S^\circ \subset M$ and $S_\infty \subset \Gamma \backslash \mathrm{Iso}(V)$, let M_S^* be the Riemann surface with the points in S removed. Suppose $\epsilon > 0$ is sufficiently small such that $B_\epsilon(\zeta)$ are all disjoint for $\zeta \in S$. Then

$$(2.5) \quad M_{S,\epsilon}^* := M^* \setminus \left(\bigsqcup_{\zeta \in S} B_\epsilon(\zeta) \right).$$

is a compact Riemann surface with boundary $\partial M_{S,\epsilon}^* = -\bigsqcup_{\zeta \in S} \partial B_\epsilon(\zeta)$.

Suppose $S = S_\infty = \Gamma \backslash \mathrm{Iso}(V) \subset M^*$, then $M = M_S^*$ and the fundamental domain of $M_{S,\epsilon}^*$ can be described as follows. For $T > 1$, let $\mathcal{F}_T := \{z \in \mathbb{H} : |\mathrm{Re}(z)| \leq 1/2 \text{ and } T \geq |z| \geq 1\}$ be the standard, truncated fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. For $\alpha \in \mathbb{N}$, denote

$$\mathcal{F}_T^\alpha := \bigcup_{j=0}^{\alpha-1} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \mathcal{F}_T.$$

Then for $\epsilon > 0$ sufficiently small, the fundamental domain of $M_{S,\epsilon}^*$ is given by

$$(2.6) \quad \mathcal{F}_T(\Gamma) := \bigcup_{\ell \in \Gamma \setminus \text{Iso}(V)} \sigma_\ell \mathcal{F}_T^{\alpha_\ell},$$

where $T = -\log \epsilon / 2\pi$.

2.3. Differentials of the third kind. Let X be a connected, non-singular projective curve over \mathbb{C} of genus g . A *differential form of the third kind* is a meromorphic differential 1-form η on X with at most simple poles and residues in \mathbb{Q} . Denote the set of its singularities by

$$(2.7) \quad S(\eta) := \{\zeta \in X : \eta \text{ has a simple pole at } \zeta\},$$

and the residue of η at $z \in X$ by $r_z(\eta)$. Given a real-analytic function h with no poles in the ϵ_0 -neighborhood $B_{\epsilon_0}(z)$ for sufficiently small $\epsilon_0 > 0$, we have

$$(2.8) \quad h(z)r_z(\eta) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} \int_{\partial B_\epsilon(z)} h\eta.$$

This is true for all $\epsilon > 0$ sufficiently small when h is holomorphic. The *residue divisor* of η is defined by

$$(2.9) \quad \text{res}(\eta) := \sum_{\zeta \in S(\eta)} r_\zeta(\eta) \cdot [\zeta] \in \text{Div}^0(X)$$

and has degree zero by the residue theorem. Conversely, for any divisor $D \in \text{Div}^0(X)$, there is a differential of the third kind η_D such that $\text{res}(\eta_D) = D$ [3, 12]. Furthermore, such a differential is uniquely determined up to addition of holomorphic differentials. In fact, by the Riemann period relations, there exists a *unique* differential of the third kind η_D for each $D \in \text{Div}^0(X)$ such that

$$\text{Re} \int_\gamma \eta_D = 0$$

for every smooth curve $\gamma \subset X \setminus D$. We call η_D the *canonical differential of the third kind*.

For a finite set $S \subset X$, denote $X' := X \setminus S$. The homology group $H_1(X', \mathbb{Z})$ is a finitely generated free abelian group, and there is a canonical surjective homomorphism

$$(2.10) \quad \pi : H_1(X', \mathbb{Z}) \twoheadrightarrow H_1(X, \mathbb{Z})$$

induced by $X' \hookrightarrow X$. The kernel is isomorphic to $\mathbb{Z}^{|S|}$ and has a basis consisting of the class of $\partial B_\epsilon(\zeta)$ in $H_1(X', \mathbb{Z})$ for each $\zeta \in S$ and $\epsilon > 0$ sufficiently small. We denote these classes by $\mathbf{c}_\zeta \in H_1(X', \mathbb{Z})$. Every $\mathbf{c}' \in \ker(\pi)$ can be expressed as a unique \mathbb{Z} -linear combinations of these \mathbf{c}_ζ 's. We define the winding number $\text{ind}(\mathbf{c}'; \zeta)$ of \mathbf{c}' around $\zeta \in S$ to be the coefficient of \mathbf{c}_ζ in this expression. By (2.8), we have

$$(2.11) \quad \frac{1}{2\pi i} \int_{\mathbf{c}'} \eta = \sum_{\zeta \in S} \text{ind}(\mathbf{c}'; \zeta) \cdot r_\zeta(\eta),$$

for any differential of the third kind η satisfying $S(\eta) \subset S$.

To define the winding number of any $\mathbf{c}' \in H_1(X', \mathbb{Z})$, we need to fix a section s of the surjective homomorphism (2.10). Then we can define $\text{ind}_s(\mathbf{c}'; \zeta)$ for any $\mathbf{c}' \in H_1(X', \mathbb{Z})$ and $\zeta \in S$ by

$$(2.12) \quad \text{ind}_s(\mathbf{c}'; \zeta) := \text{ind}(\mathbf{c}' - s \circ \pi(\mathbf{c}'); \zeta).$$

This quantity clearly depends on the section s and satisfies

$$(2.13) \quad \mathbf{c}' - s \circ \pi(\mathbf{c}') = \sum_{\zeta \in S} \text{ind}_s(\mathbf{c}'; \zeta) \mathbf{c}_\zeta.$$

in $H_1(X', \mathbb{Z})$.

For $\zeta_1, \zeta_2 \in S$, let $\gamma \subset X$ be any curve going from ζ_1 to ζ_2 and $\mathbf{I}(\gamma, \mathbf{c}')$ the signed intersection of γ with any $\mathbf{c}' \in H_1(X', \mathbb{Z})$. It can be defined as the integral of the Poincaré dual of $\mathbf{c}' \in H_1(X', \mathbb{Z})$ along γ . It is clear that $\mathbf{I}(\gamma, \mathbf{c}')$ only depends on the class of γ in $H_1(X', \partial X', \mathbb{Z})$. From the fundamental polygon of a Riemann surface, we can see that for every section s and pairs of points $\zeta_1, \zeta_2 \in S$, there exists a curve γ_s such that $\mathbf{I}(\gamma_s, s(\mathbf{c})) = 0$ for all $\mathbf{c} \in H_1(X, \mathbb{Z})$.

For each differential η of the third kind with $S(\eta) \subset S$, there exists (by Poincaré duality) a closed differential $\omega_{\eta, s}$ on X unique in $H_{\text{dR}}^1(X)$ such that

$$(2.14) \quad \int_{s(\mathbf{c})} \eta = \int_{\mathbf{c}} \omega_{\eta, s}$$

for all $\mathbf{c} \in H_1(X, \mathbb{Z})$. For any $\mathbf{c}' \in H_1(X', \mathbb{Z})$, we have

$$(2.15) \quad \frac{1}{2\pi i} \int_{\mathbf{c}'} \eta - \omega_{\eta, s} = \sum_{\zeta \in S} \text{ind}_s(\mathbf{c}'; \zeta) \cdot r_\zeta(\eta).$$

Using (2.13), we can relate the intersection number to this integral for certain η .

Lemma 2.2. *Suppose $\text{res}(\eta) = \zeta_1 - \zeta_2$. In the notation above, we have*

$$(2.16) \quad \frac{1}{2\pi i} \int_{\mathbf{c}'} \eta - \omega_{\eta, s} = -\mathbf{I}(\gamma_s, \mathbf{c}')$$

for any $\mathbf{c}' \in H_1(X', \mathbb{Z})$.

2.4. Geodesics and Cycle Integrals. An element $X \in V(\mathbb{Q})$ of positive norm m defines a geodesic $c_X \subset \mathcal{D}$ via

$$c_X := \{z \in \mathcal{D} : z \perp X\} = \{z \in \mathcal{D} : (X, X(z)) = 0\}.$$

Suppose $X = \begin{pmatrix} -B/2 & -C \\ A & B/2 \end{pmatrix}$. Then c_X is explicitly given by $\{z \in \mathbb{H}^* : A|z|^2 + B\text{Re}(z) + C = 0\}$. The geodesics carry a natural orientation given as follows. The semicircle c_X is oriented counter-clockwise if $A > 0$ and clockwise if $A < 0$. The stabilizer $\Gamma_X \subset \Gamma$ of X is either

infinite cyclic or trivial. The latter happens if and only if $\sqrt{m/N} \in \mathbb{Q}^\times$. In that case, we call X *split-hyperbolic*. For convenience, we also set

$$(2.17) \quad \iota(X) := \begin{cases} 1, & X \text{ is split-hyperbolic,} \\ 0, & \text{otherwise.} \end{cases}$$

When $\iota(X) = 1$, the orthogonal complement X in V is spanned by two rational isotropic lines $\ell_X, \tilde{\ell}_X$ and c_X connects the two corresponding cusps of \mathcal{D} . We distinguish them by requiring ℓ_X to be the endpoint of c_X . Since the action of Γ preserves the orientation of c_X , we have $\ell_{\gamma \cdot X} = \gamma \cdot \ell_X$ for all $\gamma \in \Gamma$. Note that $\tilde{\ell}_X = \ell_{-X}$. Choose $\sigma_{\ell_X} \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$(2.18) \quad \sigma_{\ell_X}^{-1} X = \sqrt{m/N} \begin{pmatrix} 1 & -2r_X \\ & -1 \end{pmatrix}$$

for a unique $r_X \in \mathbb{Q} \cap [-\alpha_{\ell_X}/2, \alpha_{\ell_X}/2)$. We call r_X the real part of c_X , since the geodesic c_X is given by $c_X = \sigma_{\ell_X} \{z \in \mathcal{D} : \mathrm{Re}(z) = r_X\}$. We denote the image of c_X on M^* by $c(X)$, which is a closed geodesic cycle when X is not split-hyperbolic.

Let η be a differential form of the third kind on M^* and $S := S(\eta) \cup (\Gamma \backslash \mathrm{Iso}(V))$. If $c(X)$ and S are disjoint, then the cycle integral $\int_{c(X)} \eta$ converges. Otherwise, we can regularize it in the following way. For $S \subset M^*$ finite and $\epsilon > 0$ sufficiently small, we write

$$c(X; S, \epsilon) := c(X) \cap M_{S, \epsilon}^*$$

which is a compact subset of M_S^* . Then we can define

$$(2.19) \quad \int_{c(X)} \eta := \mathrm{p.v.} \lim_{\epsilon \rightarrow 0} \left(\int_{c(X; S, \epsilon)} \eta - \iota(X) (r_{\ell_X}(\eta) - r_{\ell_{-X}}(\eta)) \frac{\log \epsilon}{2\pi i} \right),$$

where we take the Cauchy principal value of the limit. It exists by the proposition below.

Proposition 2.3. *The right hand side of (2.19) exists.*

Proof. If $S \cap c(X) = \emptyset$, then the limit clearly exists. Otherwise, we can understand the right hand side as the follows. Suppose $\zeta \in S^\circ(\eta) \cap c(X) \subset M$, then for any $\epsilon > 0$, $c(X) \cap \partial B_\epsilon(\zeta)$ has two points, which can be connected along $\partial B_\epsilon(\zeta)$ in two ways. We can keep $c(X)$ closed by deforming it near ζ in these two ways. Each will yield a value as $\epsilon \rightarrow 0$. The Cauchy principal value of $\lim_{\epsilon \rightarrow 0} \int_{c(X; S(\eta), \epsilon) \cap B_{\epsilon_0}(\zeta)} \eta$ is then the average of these two values. Here ϵ_0 is chosen small enough such that the $B_{\epsilon_0}(z)$'s are disjoint for $z \in S$.

When $\iota(X) = 1$, the geodesic $c(X)$ connects two cusps $\ell = \ell_X$ and ℓ_{-X} on M^* . Near $\zeta \in S^\circ(\eta) \cap c(X)$, the same argument as for $\iota(X) = 0$ works. Near the cusp ℓ , we have $\eta = (r_\ell(\eta) + O(e^{-Cy_\ell})) dz_\ell$ in the local coordinate $z_\ell = x_\ell + iy_\ell = \sigma_\ell^{-1} z$ by the definition of $r_\ell(\eta)$. Therefore, we have

$$\int_{c(X) \cap (B_{\epsilon_0}(\ell) - B_\epsilon(\ell))} \eta = r_\ell(\eta) \int_{-\log \epsilon_0/2\pi}^{-\log \epsilon/2\pi} idy_\ell + O(\epsilon) = r_\ell(\eta) \frac{\log \epsilon}{2\pi i} + O(\epsilon)$$

near the cusp $\ell = \ell_X$. The same argument with the reversed orientation also works for ℓ_{-X} and finishes the proof. \square

3. THETA SERIES AND THE MAIN RESULT

3.1. Weil representation and modular forms. Let $\text{Mp}_2(\mathbb{R})$ be the metaplectic two-fold cover of $\text{SL}_2(\mathbb{R})$ consisting of elements $(M, \phi(\tau))$ with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \rightarrow \mathbb{C}$ a holomorphic function satisfying $\phi(\tau)^2 = c\tau + d$. Let $\Gamma' \subset \text{Mp}_2(\mathbb{R})$ be the inverse image of $\text{SL}_2(\mathbb{Z}) \subset \text{SL}_2(\mathbb{R})$ under the covering map. It is generated by $T := ((\frac{1}{0} \frac{1}{1}), 1)$ and $S = ((\frac{0}{1} \frac{-1}{0}), \sqrt{\tau})$, where $\sqrt{\cdot}$ denotes the principal branch of the holomorphic square root.

Let L be an even lattice with quadratic form Q and associated bilinear form (\cdot, \cdot) and dual lattice L' of signature (b^+, b^-) . The (finite) Weil representation ρ_L of Γ' associated to L acts on $\mathbb{C}[L'/L]$ (see e.g., [2, 21]) and is given by

$$(3.1) \quad \rho_L(T)(\mathbf{e}_h) := \mathbf{e}(Q(h))\mathbf{e}_h, \quad \rho_L(S)(\mathbf{e}_h) := \frac{\mathbf{e}(-(b^+ - b^-)/8)}{\sqrt{|L'/L|}} \sum_{\mu \in L'/L} \mathbf{e}(-(h, \mu))\mathbf{e}_\mu.$$

Here we denote the standard basis of $\mathbb{C}[L'/L]$ by $\{\mathbf{e}_h : h \in L'/L\}$. Furthermore, we write $\langle \cdot, \cdot \rangle$ for the standard hermitian scalar product on $\mathbb{C}[L'/L]$. Note that if $L = L_1 \oplus L_2$ then $\mathbb{C}[L'/L]$ can be identified with $\mathbb{C}[L'_1/L_1] \otimes \mathbb{C}[L'_2/L_2]$ by sending $\mathbf{e}_{(h_1, h_2)}$ to $\mathbf{e}_{h_1} \otimes \mathbf{e}_{h_2}$ for $(h_1, h_2) \in L'_1/L_1 \oplus L'_2/L_2 = L'/L$, and we have $\rho_L = \rho_{L_1} \otimes \rho_{L_2}$.

A function $f : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is called *modular* with weight $k \in \frac{1}{2}\mathbb{Z}$, representation ρ_L and level $\Gamma \subset \Gamma'$ if

$$(3.2) \quad (f|_k(M, \phi))(\tau) := \phi(\tau)^{-2k} f(M \cdot \tau) = \rho_L((M, \phi))f(\tau)$$

for all $(M, \phi(\tau)) \in \Gamma$. Let $\mathcal{A}_k(\Gamma, \rho_L)$ denote the space of such functions that are real-analytic. It contains the usual subspaces $M_k(\Gamma, \rho_L)$, $S_k(\Gamma, \rho_L)$ of holomorphic modular and cusp forms respectively, and also $H_k(\Gamma, \rho_L)$, the space of harmonic (weak) Maass forms, see [4]. We drop Γ , resp. ρ_L , if it is Γ' , resp. trivial. In case $\Gamma \subset \text{Mp}_2(\mathbb{R})$ is the double cover of $\Gamma'' \subset \text{SL}_2(\mathbb{Z})$, we also write $\mathcal{A}_k(\Gamma'', \rho_L)$ for $\mathcal{A}_k(\Gamma, \rho_L)$. The same notation holds for any subspace.

In [27], Zagier defined the notion of a *mock modular form* following Zwegers [29]. For a cusp form $g(\tau) \in S_{2-k}(\bar{\rho}_L)$, define

$$(3.3) \quad g^*(\tau) := \frac{i}{2} \int_{-\bar{\tau}}^{i\infty} \frac{\overline{g(-\bar{z})}}{(-i(z + \tau)/2)^k} dz.$$

A holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is called *mock modular* of weight k with respect to ρ_L if there exists a $g \in S_{2-k}(\bar{\rho}_L)$ such that $\tilde{f} := f + g^* \in H_k(\rho_L)$. The form g is called the *shadow* of f . The shadow g is related to $f + g^*$ via the operator $\xi_k := 2iv^k \partial_{\bar{\tau}}$ in [4], i.e., $\xi_k(f(\tau) + g^*(\tau)) = g(\tau)$.

The notion of mock modular form was further generalized in [7] to mixed mock modular forms. We say that a holomorphic function f is *mixed mock modular* if there exists $g \in$

$S_{2-k+\ell}(\rho_{L_1})$ and $h \in M_\ell(\rho_{L_2})$ such that $f + h \otimes g^* \in \mathcal{A}_k(\Gamma, \rho_L)$. In this case, the function $g(\tau) \otimes \bar{h}(\tau)$, which is modular of weight $2 - k$ after multiplying by v^ℓ , is called the *shadow* of f .

3.2. Unary theta series associated to a cusp. Suppose $b^+ = 2, b^- = 1$ and L contains isotropic vectors. For an isotropic line $\ell \in \text{Iso}(V)$, the space ℓ^\perp/ℓ is a one-dimensional positive definite space with quadratic form Q . It contains an even lattice

$$K_\ell := (L \cap \ell^\perp)/(L \cap \ell),$$

whose dual lattice is given by $K'_\ell := (L' \cap \ell^\perp)/(L' \cap \ell)$. There is an exact sequence

$$(L' \cap \ell)/(L \cap \ell) \rightarrow (L' \cap \ell^\perp)/(L \cap \ell^\perp) \rightarrow K'_\ell/K_\ell.$$

For $h \in L'/L$ with $h \perp \ell$, let \bar{h} denote its image in K'_ℓ/K_ℓ . Define a unary theta function Θ_ℓ by

$$(3.4) \quad \Theta_\ell(\tau) := \sum_{h \in L'/L, h \perp \ell} \mathfrak{e}_h \sum_{X \in K_\ell + \bar{h}} (X, \text{Re}(W(\sigma_\ell i))) \mathfrak{e}(Q(X)\tau).$$

It transforms with weight $3/2$ and the Weil representation ρ_L . Furthermore, it only depends on the class of ℓ in $\Gamma \backslash \text{Iso}(V)$. The Fourier coefficients take the following shape.

Lemma 3.1. *For $m \in \mathbb{Q}_{>0}$ and $h \in L'/L$, let $b_\ell(m, h)$ be the $(m, h)^{\text{th}}$ Fourier coefficient of $\Theta_\ell(\tau)$. Then*

$$(3.5) \quad b_\ell(m, h) = -\frac{\sqrt{N}}{2\varepsilon_\ell} \sum_{X \in \Gamma_\ell \backslash (L_{m,h} \cap \ell^\perp)} \delta_\ell(X),$$

where $L_{m,h} := \{\lambda \in L + h : Q(\lambda) = m\}$ and $\delta_\ell(X) = \pm 1$ if $\ell = \ell_{\pm X}$.

Proof. If h is not orthogonal to ℓ , then both sides above are zero trivially. So suppose $h \perp \ell$. Then there is a natural surjection from the set $L_{m,h} \cap \ell^\perp$ to $\{X \in K_\ell + \bar{h} : Q(X) = m\}$. The kernel is the subgroup of $L' \cap \ell$ that acts on $L_{m,h} \cap \ell^\perp$ by addition, which is exactly $L \cap \ell$. The map $L_{m,h} \cap \ell^\perp \rightarrow (L_{m,h} \cap \ell^\perp)/(L \cap \ell)$ now factors through $L_{m,h} \cap \ell^\perp \rightarrow \Gamma_\ell \backslash (L_{m,h} \cap \ell^\perp)$. Arguing as in Lemma 3.7 of [8], we know that

$$\Gamma_\ell \backslash (L_{m,h} \cap \ell^\perp) \rightarrow (L_{m,h} \cap \ell^\perp)/(L \cap \ell)$$

is an $2\sqrt{m/N}\varepsilon_\ell$ to 1 covering map. For an element $X \in L_{m,h} \cap \ell^\perp$ with $\ell_X = \ell$, (2.18) implies that

$$(X, \text{Re}(W(\sigma_\ell i))) = \frac{1}{2\sqrt{N}} \left(\sigma_\ell^{-1} X, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) = -\sqrt{m},$$

with the sign changed if $\ell_{-X} = \ell$ instead. This finishes the proof. \square

3.3. Schwartz Forms and Theta functions. Recall that the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$ acts on the space of Schwartz functions on $V_{\mathbb{R}}$ via the Weil representation ω over \mathbb{R} . For a convenient reference, see e.g. [21]. We consider the Shintani Schwartz function φ_{Sh} [21],

$$(3.6) \quad \varphi_{\mathrm{Sh}}(X, z) := (X, W(z))e^{-\pi(X, X)_z}.$$

Then $\varphi_{\mathrm{Sh}}(X, z)$ has weight $3/2$, that is,

$$\omega(k'(\theta))\varphi_{\mathrm{Sh}} = \chi_{3/2}(k'(\theta))\varphi_{\mathrm{Sh}}.$$

Here $k'(\theta)$ is a preimage of $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in \mathrm{SO}(2)$ in $\mathrm{Mp}_2(\mathbb{R})$ and $\chi_{3/2}$ is the character of K' whose square is given by $\chi_{3/2}^2(k'(\theta)) = e^{3i\theta}$. The Kudla-Millson Schwartz form [16] is closely related to φ_{Sh} and is given by

$$(3.7) \quad \varphi_{\mathrm{KM}}(X, z) := \varphi_{\mathrm{Sh}}(X, z)d\bar{z} + \overline{\varphi_{\mathrm{Sh}}(X, z)}dz.$$

Then $\varphi_{\mathrm{KM}}(X)$ defines a closed 1-form on \mathcal{D} . The associated “Millson” Schwartz function [18],

$$(3.8) \quad \psi(X, z) := \frac{(X, X(z))}{2}e^{-\pi(X, X)_z}$$

has weight $-1/2$. In what follows, it will be convenient to set

$$(3.9) \quad \varphi^0(X, z) := \varphi(X, z)e^{\pi(X, X)},$$

where φ is any of the functions above. For a Schwartz function φ of weight ℓ , we set

$$\varphi(X, \tau, z) := v^{-\ell/2}\omega(g'_\tau)\varphi(x) = v^{3/4-\ell/2}\varphi^0(\sqrt{v}X, z)e^{\pi i(X, X)\tau}.$$

Here, $g'_\tau = \left(\begin{pmatrix} u^{1/2} & v^{-1/2}v \\ 0 & u^{-1/2} \end{pmatrix}, u^{-1/4} \right) \in \mathrm{Mp}_2(\mathbb{R})$ maps the base point $i \in \mathbb{H}$ to $\tau = u + iv \in \mathbb{H}$. Finally note that all functions (and forms) above are $G(\mathbb{R})$ -equivariant, that is,

$$\varphi(g \cdot X, g \cdot z) = \varphi(X, z) \quad (g \in G(\mathbb{R})).$$

In particular, for fixed X , the above functions are Γ_X -invariant.

For $\varphi \in \{\varphi_{\mathrm{Sh}}, \varphi_{\mathrm{KM}}, \psi\}$ and $L \subset V$ a lattice as in Section 2.1, we define the theta series

$$(3.10) \quad \Theta(\tau, z, \varphi) := \sum_{h \in L'/L} \Theta_h(\tau, z, \varphi)\mathfrak{e}_h, \quad \Theta_h(\tau, z, \varphi) := \sum_{X \in L+h} \varphi(X, \tau, z).$$

The series all converge absolutely due to the exponential decay of φ . Then $\Theta(\tau, z, \varphi_{\mathrm{Sh}})$ and $\Theta(\tau, z, \psi)$ transform in τ with respect to the Weil representation ρ_L with weight $3/2$ and $-1/2$ respectively, see [21]. As a function of z , $\Theta(\tau, z, \varphi_{\mathrm{Sh}})$ and $\Theta(\tau, z, \psi)$ have weights 2 and 0 for Γ_L respectively, while $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ defines a closed differential 1-form on M .

When $m \in \mathbb{Q}^\times$, the set $\Gamma \backslash L_{m,h}$ is finite, where $L_{m,h}$ is defined in Lemma 3.1. Therefore, we can write the Fourier expansion of $\Theta_h(\tau, z, \varphi)$ as (see [5, Equation (4.15)])

$$(3.11) \quad \Theta_h(\tau, z, \varphi) = \Theta_{0,h}(v, z, \varphi^0) + \sum_{m \in \mathbb{Q}^\times} \mathfrak{e}(m\tau) \sum_{X \in \Gamma \backslash L_{m,h}} a(\sqrt{v}X, z, \varphi^0),$$

where

$$(3.12) \quad \Theta_{0,h}(v, z, \varphi^0) := \sum_{X \in L_{0,h}} \varphi^0(\sqrt{v}X, z), \quad a(X, z, \varphi^0) := \sum_{\gamma \in \Gamma_X \backslash \Gamma} \varphi^0(\gamma^{-1}X, z).$$

Using a partial Poisson summation, we can describe their behaviors near the cusps of M as follows (see [11]).

Proposition 3.2. *Let $z_\ell = x_\ell + iy_\ell := \sigma_\ell^{-1}z$ be the local coordinate at the cusp $\ell \in \text{Iso}(V)$. As $y_\ell \rightarrow \infty$, we have*

$$(3.13) \quad \begin{aligned} \Theta(\tau, z, \varphi_{\text{Sh}})dz &= \left(\frac{1}{\sqrt{N}\beta_\ell} \Theta_\ell(\tau) + O(e^{-Cy_\ell^2 \min(v, 1/v)}) \right) dz_\ell, \\ \Theta(\tau, z, \psi) &= O(e^{-Cvy_\ell^2}) \end{aligned}$$

for some absolute constant $C > 0$ depending only on L . Furthermore, let $X \in L_{m,h}$ be a hyperbolic element, i.e. $m > 0$. Then

$$(3.14) \quad a(\sqrt{v}X, z, \varphi_{\text{Sh}}^0) = \begin{cases} \left(\mp \frac{1}{2\alpha_\ell} + O(e^{-Cy_\ell^2 \min\{mv, 1/(mv)\}}) \right) & \text{if } \iota(X) = 1, [\ell] = [\ell_{\pm X}], \\ O(e^{-Cmvy_\ell^2}), & \text{otherwise.} \end{cases}$$

Here α_ℓ is the width of the cusp ℓ as defined in Section 2.2.

We can extend $\Theta(\tau, z, \varphi_{\text{Sh}})$ and $\Theta(\tau, z, \psi)$ to M^* by continuity.

3.4. A singular Schwartz function. We define a singular (Schwartz) function $\tilde{\psi}$ on $V_{\mathbb{R}}$ by

$$(3.15) \quad \tilde{\psi}(X, z) := -\frac{\text{sgn}(X, X(z))}{2} \text{erfc}(\sqrt{\pi}|(X, X(z))|) e^{-\pi(X, X)},$$

where $\text{erfc}(t) := \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-r^2} dr$ is the complementary error function, and $\text{sgn}(x)$ is the sign function with the convention $\text{sgn}(0) = 0$. We set $\tilde{\psi}(0) = 0$. Note that $\tilde{\psi}(X, z)$ has a singularity along the geodesic c_X if $Q(X) > 0$ and is smooth otherwise. We also set $\tilde{\psi}(X, z) = 0$ along c_X . As before we write $\tilde{\psi}^0(X, z) = \tilde{\psi}(X, z)e^{\pi(X, X)}$ and set

$$\tilde{\psi}(X, \tau, z) = \tilde{\psi}^0(\sqrt{v}X, z) \mathbf{e}(Q(X)\tau)$$

as if $\tilde{\psi}$ had weight $3/2$ under the Weil representation (which of course it doesn't). This is motivated by the following Lemma 3.3(ii) which can be obtained by a direct calculation (such as in [9]).

Lemma 3.3. (i) *Away from the singularity of $\tilde{\psi}^0(X, z)$,*

$$(3.16) \quad \bar{\partial} \tilde{\psi}^0(X, z) = \varphi_{\text{Sh}}^0(X, z) d\bar{z}.$$

Here $\bar{\partial}$ is the exterior anti-holomorphic derivative for $z \in \mathcal{D}$. In particular, for $Q(X) \leq 0$, the function $\tilde{\psi}^0(X, z)$ is a smooth $\bar{\partial}$ -primitive of $\varphi_{\text{Sh}}^0(X, z)d\bar{z}$. When $Q(X)$ is positive, the function $\tilde{\psi}^0(X, z)$ satisfies the following current equation,

$$(3.17) \quad \int_{\Gamma_X \setminus \mathcal{D}} (\bar{\partial}\tilde{\eta}) \tilde{\psi}^0(X, z) = - \int_{c_X} \tilde{\eta} + \int_{\Gamma_X \setminus \mathcal{D}} \tilde{\eta} \wedge \varphi_{\text{Sh}}^0(X, z)d\bar{z}.$$

for any compactly supported 1-form $\tilde{\eta}$ on $\Gamma_X \setminus \mathcal{D}$.

(ii) For the action of $L_\tau := -2iv^2\partial_{\bar{\tau}}$, the weight-lowering operator in τ , we have

$$(3.18) \quad L_\tau \tilde{\psi}(X, \tau, z) = \psi(X, \tau, z).$$

We do not introduce the “full” theta series $\Theta(\tau, z, \tilde{\psi})$ since it would yield a function with dense singularities in \mathcal{D} . However, the individual Fourier coefficients define much better behaved functions with locally finite singularities. We set

$$(3.19) \quad \Theta_{m,h}(v, z, \tilde{\psi}^0) := \sum_{X \in L_{m,h}} \tilde{\psi}^0(\sqrt{v}X, z) \quad (m \in \mathbb{Q})$$

and also write if $Q(X) \neq 0$,

$$(3.20) \quad a(X, z, \tilde{\psi}^0) := \sum_{\gamma \in \Gamma_X \setminus \Gamma} \tilde{\psi}^0(\gamma^{-1}X, z)$$

for the individual Γ -orbits.

We now state the behavior of $\Theta_{m,h}(v, z, \tilde{\psi})$ at the cusps, which will be proved in the next subsection. We let $\mathbb{B}_1(x)$ be the first periodic Bernoulli polynomial given by

$$(3.21) \quad \mathbb{B}_1(x) := x - (\lceil x \rceil + \lfloor x \rfloor)/2.$$

Lemma 3.4. *Let $X \in L_{m,h}$ be a hyperbolic element, i.e., $m > 0$, and let $\ell \in \text{Iso}(V)$ be a cusp with local coordinate $z_\ell := x_\ell + iy_\ell = \sigma_\ell^{-1}z$. The function $a(\sqrt{v}X, z, \tilde{\psi}^0)$ has the following asymptotic expansions at ℓ ,*

$$(3.22) \quad a(\sqrt{v}X, z, \tilde{\psi}^0) = \begin{cases} \mp \mathbb{B}_1\left(\frac{x_\ell - r_{\pm X}}{\alpha_\ell}\right) + O(e^{-C y_\ell^2 \min\{mv, 1/(mv)\}}), & \text{if } \iota(X) = 1, [\ell] = [\ell_{\pm X}], \\ O(e^{-C m v y_\ell^2}), & \text{otherwise.} \end{cases}$$

Here r_X is defined in (2.18). Let β_ℓ, h_ℓ be the quantities defined in Section 2.2. Then

$$\Theta_{0,h}(v, z, \tilde{\psi}^0) = -\mathbb{B}_1\left(\frac{h_\ell}{\beta_\ell}\right) + O(e^{-C v y_\ell^2}) \quad (y_\ell \rightarrow \infty).$$

We set for all $h \in L'/L$ and $[\ell] \in \Gamma \setminus \text{Iso}(V)$,

$$(3.23) \quad \Theta_{m,h}(v, [\ell], \tilde{\psi}^0) := \begin{cases} 0 & m \neq 0, \\ -\mathbb{B}_1\left(\frac{h_\ell}{\beta_\ell}\right) & m = 0. \end{cases}$$

3.5. Boundary Behavior of $\tilde{\psi}$. For $\operatorname{Re}(s) > -1/2$ and $\kappa > 0$, define

$$(3.24) \quad g(w; \kappa, s) := \frac{\pi^{-s-1/2}|w|^{-2s}\operatorname{sgn}(w)}{2}\Gamma\left(\pi\kappa w^2, s + \frac{1}{2}\right).$$

Note that $g(w; \kappa, 0) = \frac{\operatorname{sgn}(w)}{2}\operatorname{erfc}(\sqrt{\pi\kappa}|w|)$ and $g(w; \kappa, 1/2) = \frac{e^{-\pi w^2\kappa}}{2\pi w}$. We are interested in the periodic function

$$(3.25) \quad G(x; \kappa, s) := \sum_{n \in \mathbb{Z}} g(x + n; \kappa, s),$$

which converges absolutely and uniformly on compact subsets of $\mathbb{R} \setminus \mathbb{Z}$ for $\operatorname{Re}(s) > -1/2$. Using Poisson summation, we can calculate its Fourier expansion.

Lemma 3.5. *At $s = 0$, the function $G(x; \kappa, s)$ is bounded, 1-periodic in x and has the expansion*

$$(3.26) \quad G(x; \kappa, 0) = -\mathbb{B}_1(x) + i \sum_{m \in \mathbb{Z}, m \neq 0} g(m, \kappa^{-1}, 1/2)\mathbf{e}(mx),$$

where the sum above converges absolutely and uniformly.

Proof. It is easy to see that the function $h(x, s) := G(x; \kappa, s) + \mathbb{B}_1(x)$ is bounded, odd and 1-periodic on \mathbb{R} for $0 \geq \operatorname{Re}(s) > -1/2$. Therefore, we just need to calculate its m^{th} Fourier coefficients for $m \neq 0$. In this case, we start with

$$\int_0^1 h(x, s)\mathbf{e}(-mx)dx = \widehat{g}(-m; \kappa, s) - \frac{1}{2\pi im},$$

which becomes the m^{th} Fourier coefficient of $h(x, 0)$ after taking $s \rightarrow 0$ from the left by the dominated convergence theorem. A standard calculation gives us

$$\widehat{g}(\widehat{w}; \kappa, s) := \int_{-\infty}^{\infty} g(w; \kappa, s)\mathbf{e}(w\widehat{w})dw = i \left(\frac{\operatorname{sgn}(\widehat{w})}{|\widehat{w}|^{1-2s}} \frac{\Gamma(1-s)}{2\pi^{1-s}} - g(\widehat{w}, \kappa^{-1}, 1/2-s) \right)$$

for $-1/2 < \operatorname{Re}(s) < 1$. Taking $s \rightarrow 0$ and using the oddness of g then gives us (3.26). \square

Proof of Lemma 3.4. Suppose $\iota(X) = 1$ and $\ell = \ell_X$. Then $m = Nk^2$ for some $k \in \mathbb{Q}_{>0}$ and Γ_X is trivial. Write $\sigma_\ell^{-1}X = k \begin{pmatrix} 1 & -2r \\ & -1 \end{pmatrix}$ with $r = r_X \in \mathbb{Q}$ as in (2.18), $\alpha = \alpha_\ell$. Arguing as in Lemma 5.2 of [5] gives us

$$a(\sqrt{v}X, z, \tilde{\psi}^0) = \sum_{\gamma \in \Gamma_\ell} \tilde{\psi}^0(\sqrt{v}X, \gamma z) + O(e^{-Cmvy_\ell^2}).$$

Using (2.4), we can rewrite

$$\sum_{\gamma \in \Gamma_\ell} \tilde{\psi}^0(\sqrt{v}X, \gamma z) = \sum_{n \in \mathbb{Z}} \tilde{\psi}^0(\sqrt{vm/N} \begin{pmatrix} 1 & 2(-r+\alpha n) \\ & -1 \end{pmatrix}, z_\ell) = G\left(\frac{x_\ell - r}{\alpha}; \frac{4mv\alpha^2}{y_\ell^2}, 0\right).$$

Applying Lemma 3.5 then finishes the proof for $\ell = \ell_X$. The case with $\ell = \ell_{-X}$ is similar. \square

The asymptotic behavior of $\Theta_{0,h}(\tau, z, \tilde{\psi})$ can be calculated analogously.

3.6. Theta Integral and Main Theorem. In this section, we will state the main theorem, which evaluates the integral in (1.1) explicitly when η is a meromorphic differential of the third kind on M^* . For a finite set $S \subset M^*$ containing $S(\eta)$, we define

$$(3.27) \quad I(\tau, \eta) := \lim_{\epsilon \rightarrow 0} \left(\int_{M_{S,\epsilon}^*} \eta \wedge \Theta(\tau, z, \varphi_{\text{Sh}}) d\bar{z} + \sum_{\ell \in \Gamma \setminus \text{Iso}(V)} \frac{r_\ell(\eta) \varepsilon_\ell}{\sqrt{N}} \Theta_\ell(\tau) \frac{\log \epsilon}{\pi i} \right).$$

Note that the second term is present only when η is not rapidly decaying at the cusps, i.e., $S_\infty(\eta) \neq \emptyset$. This agrees with the truncation in the usual parameter $T = -\frac{\log \epsilon}{2\pi}$ (see Section 2.2).

Proposition 3.6. *The limit in (3.27) exists and defines a real-analytic modular form that transforms with respect to the Weil representation ρ_L of weight $3/2$.*

Proof. Since η has only simple poles on M , no regularization is necessary when integrating away from the cusps. Then we can replace $M_{S,\epsilon}^*$ with its fundamental domain in (2.6) to obtain

$$\int_{M_{S,\epsilon}^*} \eta \wedge \Theta(\tau, z, \varphi_{\text{Sh}}) d\bar{z} = \sum_{\ell \in \Gamma \setminus \text{Iso}(V)} \int_{\sigma_\ell \mathcal{F}_T^{\alpha_\ell}} \eta \wedge \Theta(\tau, z, \varphi_{\text{Sh}}) d\bar{z}.$$

Near the cusp given by $\ell \in \text{Iso}(V)$, one can change to the local coordinate z_ℓ as in Prop. 3.2 and use the asymptotic expansions of $\Theta(\tau, z, \varphi_{\text{Sh}}) dz$ and η there to write

$$\int_{\sigma_\ell \mathcal{F}_T^{\alpha_\ell}} \eta \wedge \Theta(\tau, z, \varphi_{\text{Sh}}) d\bar{z} = \int_{\substack{0 \leq \text{Re}(z_\ell) \leq \alpha_\ell \\ 1 \leq \text{Im}(z_\ell) \leq T}} \frac{r_\ell(\eta)}{\sqrt{N} \beta_\ell} \Theta_\ell(\tau) dz_\ell \wedge d\bar{z}_\ell + O(1) = -\frac{r_\ell(\eta) \varepsilon_\ell}{\sqrt{N}} \Theta_\ell(\tau) \frac{\log \epsilon}{\pi i} + O(1),$$

which cancels the growth of the sum, and the limit in ϵ exists. Since the integral over $M_{S,\epsilon}^*$ is modular of weight $3/2$ as a function in τ for any $\epsilon > 0$ sufficiently small, so is $I(\tau, \eta)$. \square

For $m \in \mathbb{Q}$ and $h \in L'/L$, we define the m^{th} trace of η by

$$(3.28) \quad \text{Tr}_{m,h}(\eta) := \begin{cases} \frac{1}{2\pi i} \sum_{X \in \Gamma \setminus L_{m,h}} \int_{c(X)} \eta, & m > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The main result is as follows.

Theorem 3.7. *Let L be an lattice as in Section 2.1 and η be a meromorphic differential of the third kind on $M^* = M_L^*$ with residue divisor $\sum_{\zeta \in M^*} r_\zeta(\eta) \cdot [\zeta]$. Then the components of the vector-valued modular form $I(\tau, \eta) \in \mathcal{A}_{3/2}(\rho_L)$ have the expansion*

$$\frac{1}{2\pi i} I_h(\tau, \eta) = \sum_{m \in \mathbb{Q}_{>0}} \text{Tr}_{m,h}(\eta) q^m - \sum_{\zeta \in M^*} r_\zeta(\eta) \sum_{m \in \mathbb{Q}} \Theta_{m,h}(v, \zeta, \tilde{\psi}^0) q^m.$$

Under the lowering operator L_τ , we have

$$L_\tau I(\tau, \eta) = -2\pi i \Theta(\tau, \text{res}(\eta), \psi).$$

3.7. Orbital integrals and Proof of Main Theorem. Let η be a fixed meromorphic 1-form of the third kind and set $S := S(\eta) \cup (\Gamma \setminus \text{Iso}(V)) \subset M^*$.

We first consider the non-constant coefficients. Let $X \in \Gamma \setminus L_{m,h}$ with $m \neq 0$ and consider for $\epsilon > 0$, the orbital integral

$$(3.29) \quad I_X(v, \eta; S, \epsilon) := \int_{M_{S,\epsilon}^*} \eta \wedge a(\sqrt{v}X, z, \varphi_{\text{Sh}}^0) d\bar{z},$$

where $a(X, z, \varphi_{\text{Sh}}^0)$ is defined in (3.12). We directly see

Proposition 3.8. *The integral defining $I_X(v, \eta; S, \epsilon)$ converges for all $m \in \mathbb{Q}^\times$, and even termwisely when $\sqrt{Nm} \notin \mathbb{Q}$. Furthermore, we have*

$$(3.30) \quad I_X(v, \eta; S, \epsilon) = \int_{c(X; S, \epsilon)} \eta + \int_{\partial M_{S,\epsilon}^*} a(\sqrt{v}X, z, \tilde{\psi}^0) \eta.$$

Proof. The convergence follows from the same argument in [5]. The rest is simply an application of Lemma 3.3 and Stokes' theorem. \square

Proposition 3.9. *Suppose $X \in L_{m,h}$ with $m \neq 0$. Then*

$$(3.31) \quad \lim_{\epsilon \rightarrow 0} \left(I_X(v, \eta; S, \epsilon) - \iota(X)(r_{\ell_X}(\eta) - r_{\ell_{-X}}(\eta)) \frac{\log \epsilon}{2\pi i} \right) = \left(\int_{c(X)} \eta \right) - 2\pi i a(\sqrt{v}X, \text{res}(\eta), \tilde{\psi}^0),$$

where $\iota(X)$ and $\int_{c(X)} \eta$ are defined in (2.17) and (2.19) respectively.

Proof. We first subtract $\iota(X)(r_{\ell_X}(\eta) - r_{\ell_{-X}}(\eta)) \frac{\log \epsilon}{2\pi i}$ from both sides of (3.30). Taking the limit $\epsilon \rightarrow 0$ on the right hand side gives $2\pi i a(\sqrt{v}X, \text{res}(\eta), \tilde{\psi}^0) - \int_{c(X)} \eta$ by (2.8), (2.19) and $\partial M_{S,\epsilon}^* = -\bigsqcup_{\zeta \in S} \partial B_\epsilon(\zeta)$ for $\epsilon > 0$ small.

Now for $\zeta \in M$ and $\epsilon > 0$ sufficiently small, the integral $\int_{B_\epsilon(\zeta)} \eta \wedge a(\sqrt{v}X, z, \varphi_{\text{Sh}}^0) d\bar{z}$ converges since η has at most a simple pole at ζ . If $\iota(X) = 0$, then $c(X)$ is compact in M and the limit in ϵ exists. Therefore, it suffices to consider $\iota(X) = 1$ and show that the following limit exists

$$(3.32) \quad \lim_{\epsilon \rightarrow 0} \int_{\mathcal{F}_T(\Gamma) - \mathcal{F}_{T_0}(\Gamma)} \eta \wedge a(\sqrt{v}X, z, \varphi_{\text{Sh}}^0) d\bar{z} - (r_{\ell_X}(\eta) - r_{\ell_{-X}}(\eta)) \frac{\log \epsilon}{2\pi i},$$

where $T = -\log \epsilon / 2\pi$ and $T_0 > 1$ is a fixed constant depending on η and Γ . We can now decompose $\mathcal{F}_T(\Gamma) - \mathcal{F}_{T_0}(\Gamma) = \bigsqcup_{[\ell] \in \Gamma \setminus \text{Iso}(V)} \sigma_\ell(\mathcal{F}_T - \mathcal{F}_{T_0})$. Recall that $z_\ell = x_\ell + iy_\ell = \sigma_\ell^{-1} z$

and $dz_\ell \wedge d\bar{z}_\ell = -2idx_\ell \wedge dy_\ell$. By Lemma 3.4 below, we have

$$\int_{\mathcal{F}_T - \mathcal{F}_{T_0}} \eta \wedge a(\sqrt{v}X, z_\ell, \varphi_{\text{Sh}}^0) d\bar{z}_\ell = \mp \frac{1}{2\alpha_\ell} \int_{T_0}^T \int_0^{\alpha_\ell} r_\ell(\eta) (-2i) dx_\ell dy_\ell + O(1) = \pm i r_\ell(\eta) T + O(1)$$

when $[\ell] = [\ell_{\pm X}]$, and $O(1)$ otherwise. Adding these over all $[\ell] \in \Gamma \backslash \text{Iso}(V)$ shows that the limit in (3.32) exists. \square

By the decomposition (3.11), this takes care of the non-constant coefficients of $I(\tau, \eta)$ since

$$- \sum_{\ell \in \Gamma \backslash \text{Iso}(V)} \frac{2r_\ell(\eta) \varepsilon_\ell}{\sqrt{N}} \Theta_\ell(\tau) = \sum_{h \in L'/L} \mathfrak{e}_h \sum_{m \in \mathbb{Q}^\times} \mathbf{e}(m\tau) \sum_{X \in \Gamma \backslash L_{m,h}} \iota(X) (r_{\ell_X}(\eta) - r_{\ell_{-X}}(\eta)).$$

Indeed, this is a direct consequence of Lemma 3.1 as

$$\begin{aligned} - \sum_{[\ell] \in \Gamma \backslash \text{Iso}(V)} \frac{2r_\ell(\eta) \varepsilon_\ell b_\ell(m, h)}{\sqrt{N}} &= \sum_{[\ell] \in \Gamma \backslash \text{Iso}(V)} \sum_{X \in \Gamma_\ell \backslash (L_{m,h} \cap \ell^\perp)} r_\ell(\eta) \delta_\ell(X) \\ &= \sum_{X \in \Gamma \backslash L_{m,h}} \iota(X) (r_{\ell_X}(\eta) - r_{\ell_{-X}}(\eta)). \end{aligned}$$

From the definition of $I(\tau, \eta)$ in (3.27), the constant term is given by

$$I_{0,h}(\tau, \eta) := \lim_{\epsilon \rightarrow 0} \int_{M_{S,\epsilon}^*} \eta \wedge \Theta_{0,h}(v, z, \varphi_{\text{Sh}}^0) d\bar{z}.$$

Applying Stokes' Theorem as in Prop. 3.8 and Lemma 3.4 we obtain

$$\begin{aligned} I_{0,h}(\tau, \eta) &= \lim_{\epsilon \rightarrow 0} \int_{\partial M_{S,\epsilon}^*} \Theta_{0,h}(v, z, \tilde{\psi}^0) \eta \\ &= - \sum_{\zeta \in M} r_\zeta(\eta) \sum_{m \in \mathbb{Q}} \Theta_{0,h}(v, \zeta, \tilde{\psi}^0) - \lim_{\epsilon \rightarrow 0} \sum_{[\ell] \in \Gamma \backslash \text{Iso}(V)} \int_{\partial B_\epsilon([\ell])} \Theta_{0,h}(v, z, \tilde{\psi}^0) \eta \\ &= - \sum_{\zeta \in M} r_\zeta(\eta) \sum_{m \in \mathbb{Q}} \Theta_{0,h}(v, \zeta, \tilde{\psi}^0) + \sum_{[\ell] \in \Gamma \backslash \text{Iso}(V)} r_{[\ell]}(\eta) \mathbb{B}_1 \left(\frac{h_\ell}{\beta_\ell} \right) \\ &= - \sum_{\zeta \in M^*} r_\zeta(\eta) \sum_{m \in \mathbb{Q}} \Theta_{0,h}(v, \zeta, \tilde{\psi}^0). \end{aligned}$$

This completes the proof of Theorem 3.7.

3.8. Proof of Theorem 1.2. For $N \in \mathbb{N}$, let $L \subset V$ be the lattice in Example 2.1. Then $\Gamma = \Gamma_0(N) = \Gamma_L \cap \mathrm{SO}^+(L)$ and we have a bijection

$$(3.33) \quad \bigcup_{h \in L'/L} L_{m,h} \rightarrow \mathcal{Q}_{N,4Nm}$$

$$\begin{pmatrix} -B/(2N) & -C/N \\ A & B/(2N) \end{pmatrix} \mapsto [NA, B, C],$$

where $\mathcal{Q}_{N,4Nm}$ is defined as in the introduction. Note that for any $f(\tau) = \sum_{h \in L'/L} f_h(\tau) \in \mathcal{A}_k(\rho_L)$, the component $f_0(\tau)$ is in $\mathcal{A}_k(\Gamma_0(4N))$. Applying the Fricke involution to $f_0(\tau)$ and the transformation formula of $\rho_L(S)$ tells us that $\mathrm{sc}(f(4N\tau)) \in \mathcal{A}_k(\Gamma_0(4N))$, where

$$(3.34) \quad \mathrm{sc} : \mathbb{C}[L'/L] \rightarrow \mathbb{C}$$

$$\sum_{h \in L'/L} a_h \mathbf{e}_h \mapsto \sum_{h \in L'/L} a_h.$$

Fix a fundamental discriminant $\Delta < 0$ and $r \in \mathbb{Z}$ such that $\Delta \equiv r^2 \pmod{4N}$. We follow [3] to define the genus character χ_Δ on $X = \begin{pmatrix} -B/2N & -C/N \\ A & B/2N \end{pmatrix} \in L'$ by

$$(3.35) \quad \chi_\Delta(X) = \chi_\Delta([NA, B, C]) := \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \Delta \mid B^2 - 4NAC \text{ and } (B^2 - 4NAC)/\Delta \text{ is} \\ & \text{a square modulo } 4N \text{ and } \gcd(A, B, C, \Delta) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where n is any integer prime to Δ represented by one of the quadratic forms $[N_1A, B, N_2C]$ with $N_1N_2 = N$ and $N_1, N_2 > 0$ (see also [13, §1.2] and [22, §1]). It is invariant under the action of $\Gamma_0(N)$ and all Atkin-Lehner operators.

Let $L_\Delta := (\Delta L, \frac{Q}{|\Delta|})$ the scaled lattice. Then $L'_\Delta = L'$ and there is a natural projection map $\pi : L'/L_\Delta \rightarrow L'/L$. The following group

$$(3.36) \quad \Gamma_\Delta := \Gamma_{L_\Delta} \cap \mathrm{SO}^+(L_\Delta) \subset \Gamma = \Gamma_0(N)$$

has finite index and contains the congruence subgroup $\Gamma(|\Delta|) \cap \Gamma_0(N|\Delta|)$. The linear map

$$(3.37) \quad \psi_{\Delta,r} : \mathbb{C}[L'/L] \rightarrow \mathbb{C}[L'/L_\Delta]$$

$$\mathbf{e}_h \mapsto \frac{1}{[\Gamma : \Gamma_\Delta]} \sum_{\substack{\delta \in L'/L_\Delta \\ \pi(\delta) = rh \\ \frac{Q(\delta)}{\Delta} \equiv Q(h) \pmod{\mathbb{Z}}}} \chi_\Delta(\delta) \mathbf{e}_\delta$$

intertwines the representations ρ_L and ρ_{L_Δ} [1]. With respect to the pairings on $\mathbb{C}[L'/L]$ and $\mathbb{C}[L'/L_\Delta]$, we obtain a linear map $\psi_{\Delta,r}^* : \mathcal{A}_k(\rho_{L_\Delta}) \rightarrow \mathcal{A}_k(\rho_L)$.

Now, we can apply Theorem 3.7 to the lattice L_Δ . From the bijection (3.33), we have

$$\text{sc} \circ \psi_{\Delta,r}^* \left(\sum_{\delta \in L'/L_\Delta} \text{Tr}_{m,\delta}(\eta) \mathbf{e}_\delta \right) = \text{Tr}_{N,\Delta,4Nm}(\eta)$$

for $m > 0$, where $\text{Tr}_{N,\Delta,d}$ was defined in (1.4). Similarly, the Fourier coefficients of the non-holomorphic part becomes

$$\text{sc} \circ \psi_{\Delta,r}^* \left(\sum_{\delta \in L'/L_\Delta} \Theta_{m,\delta}(v, \zeta, \tilde{\psi}^0) \mathbf{e}_\delta \right) = - \sum_{X \in \mathcal{Q}_{N,4N|\Delta|m}} \frac{\chi_\Delta(X) \text{sgn}(\text{d}(X, \zeta))}{2[\Gamma : \Gamma_\Delta]} \text{erfc} \left(\sqrt{\frac{\pi v}{N|\Delta|}} |\text{d}(X, \zeta)| \right)$$

for all nonzero $m \in \mathbb{Q}$, where we have denoted

$$(3.38) \quad \text{d}(X, \zeta) := \frac{A|\zeta|^2 + B\text{Re}(\zeta) + C}{\text{Im}(\zeta)}$$

for $\zeta \in \mathbb{H}$ and $X = [A, B, C] \in \mathbb{Z}^3$. When $m = 0$, the coefficient $\Theta_{m,h}(v, \zeta, \tilde{\psi}^0) = 0$ unless $\zeta = [\ell]$ is a cusp, in which case

$$\text{sc} \circ \psi_{\Delta,r}^* \left(\sum_{\delta \in L'/L_\Delta} \Theta_{0,\delta}(v, [\ell], \tilde{\psi}^0) \mathbf{e}_\delta \right) = - \frac{1}{[\Gamma : \Gamma_\Delta]} \sum_{\delta \in L'/L_\Delta} \chi_\Delta(\delta) \mathbb{B}_1 \left(\frac{\delta_\ell}{|\Delta| \beta_\ell} \right).$$

The quantities $\chi_\Delta(\delta)$ and β_ℓ only depends the $\Gamma_0(N)$ -class of δ and ℓ . Even though δ_ℓ depends on the choice of ℓ , the averaged quantity above does not since $\Gamma_0(N)/\Gamma_\Delta$ induces an automorphism on L'/L_Δ . If $I(\tau, \eta)$ is the integral defined in (3.27) for the lattice L_Δ , then Theorem 3.7 implies that

$$I_{\Delta,N}(\tau, \eta) := \text{sc} \circ \psi_{\Delta,r}^* \left(\frac{I(4N\tau, \eta)}{2\pi i} \right) = \sum_{d>0, d \equiv 0,3 \pmod{4}} \text{Tr}_{N,\Delta,d}(\eta) q^d + \Theta_\Delta^*(\tau, \eta) \in \mathcal{A}_{3/2}(\Gamma_0(4N)),$$

where

$$(3.39) \quad \begin{aligned} \Theta_\Delta^*(\tau, \eta) := & \sum_{\zeta \in \Gamma_0(N) \setminus \mathbb{H}} r_\zeta(\eta) \sum_{d \equiv 0,3 \pmod{4}} q^d \sum_{X \in \mathcal{Q}_{N,-\Delta d}} \frac{\chi_\Delta(X) \text{sgn}(\text{d}(X, \zeta))}{2} \text{erfc} \left(\sqrt{\frac{4\pi v}{|\Delta|}} |\text{d}(X, \zeta)| \right) \\ & + \sum_{[\ell] \in \Gamma_0(N) \setminus \mathbb{P}^1(\mathbb{Q})} r_{[\ell]}(\eta) \sum_{\delta \in L'/L_\Delta} \chi_\Delta(\delta) \mathbb{B}_1 \left(\frac{\delta_\ell}{|\Delta| \beta_\ell} \right). \end{aligned}$$

Under the lowering operator L_τ , the holomorphic part of $I_{\Delta,N}(\tau, \eta)$ vanishes and the non-holomorphic part becomes

(3.40)

$$\Theta_\Delta(\tau, \eta) := -\sqrt{\frac{v^3}{|\Delta|}} \sum_{\zeta \in \Gamma_0(N) \backslash \mathbb{H}} r_\zeta(\eta) \sum_{d \equiv 0,3 \pmod{4}} q^d \sum_{X \in \mathcal{Q}_{N,-\Delta d}} \chi_\Delta(X) d(X, \zeta) e^{-4\pi v d(X, \zeta)^2 / |\Delta|}.$$

When $N = 1$ and $\eta = \frac{j'(z)}{j(z)-1728} dz$, the series above simplifies to $v^{3/2} \overline{\theta_\Delta(\tau)}$ with θ_Δ the Siegel theta function in Theorem 1.1. The constant term of $\Theta_\Delta^*(\tau, \eta)$ can also be simplified. Since $N = 1$, $\Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q})$ contains just the cusp $\ell = \ell_0 = \mathbb{R} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, where η has residue 1. The constant term then boils down to the sum

$$\sum_{\delta \in L'/L_\Delta} \chi_\Delta(\delta) \mathbb{B}_1 \left(\frac{\delta_\ell}{|\Delta|} \right).$$

For $\delta \in L'/L_\Delta$, $\ell \cap (L_\Delta + \delta)$ is non-trivial if and only if δ represents $\begin{pmatrix} 0 & -C \\ 0 & 0 \end{pmatrix}$, in which case $\delta_\ell = -C$ and $\chi_\Delta(\delta) = \chi_\Delta([0, 0, C]) = \left(\frac{\Delta}{C}\right)$. Therefore, the sum above becomes $\sum_{C \in \mathbb{Z}/|\Delta|\mathbb{Z}} \left(\frac{\Delta}{C}\right) \mathbb{B}_1 \left(\frac{-C}{|\Delta|}\right) = L(0, \Delta)$ by Theorem 4.2 in [25].

4. INDEFINITE THETA FUNCTIONS

In his thesis [29], Zwegers gave three ways to complete the mock theta functions of Ramanujan into a real-analytic modular object. One of the ways is through the use a real-analytic theta function $\vartheta_L^{c_1, c_2}(\tau) = \sum_{h \in L'/L} \mathbf{e}_h \vartheta_{L,h}^{c_1, c_2}(\tau)$ constructed from a lattice L of signature $(p, 1)$ and two negative vectors c_1, c_2 such that $(c_1, c_2) < 0$. Then $\vartheta_L^{c_1, c_2}(\tau)$ is a real-analytic modular form of weight $(p+1)/2$ on $\text{Mp}_2(\mathbb{Z})$ with respect to the Weil representation ρ_L .

In [9] Theorem 4.7, Kudla und the second author gave another interpretation of $\vartheta_L^{c_1, c_2}$ as the integral of the Kudla-Millson theta form $\Theta(\tau, z, \varphi_{\text{KM}})$ for signature $(p, 1)$. Namely, the conditions on c_1 and c_2 imply that they define points $[c_1]$ and $[c_2]$ in the same component of the symmetric space associated to $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Then for the geodesic

$$\mathcal{D}_{c_1, c_2} = \{\text{span}(tc_1 + (1-t)c_2) : t \in [0, 1]\}$$

in the hyperbolic space \mathcal{D} connecting the points $[c_1]$ and $[c_2]$, we have

$$(4.1) \quad \vartheta_{L,h}^{c_1, c_2}(\tau) = \int_{\mathcal{D}_{c_1, c_2}} \Theta_h(\tau, z, \varphi_{\text{KM}}).$$

Furthermore,

$$(4.2) \quad \vartheta_{L,h}^{c_1,c_2}(\tau) = \sum_{m \geq 0} \sum_{X \in L_{m,h}} \frac{\text{sgn}((X, c_1)) - \text{sgn}((X, c_2))}{2} q^m \\ + \sum_{m \in \mathbb{Q}} (\Theta_{m,h}(\tau, [c_2], \tilde{\psi}) - \Theta_{m,h}(\tau, [c_1], \tilde{\psi})) q^m.$$

Here $\tilde{\psi}$ is the singular form defined in (3.15) (extended to signature $(p, 1)$). Moreover, the holomorphic coefficients $\frac{1}{2}(\text{sgn}(X, c_1) - \text{sgn}(X, c_2))$ are the signed intersection number of \mathcal{D}_{c_1,c_2} with the cycle c_X . Combining the considerations in [9] with those in [11] (or using Section 3.5) one sees that (4.1) extends to the case when c_1 or c_2 (or both) are rational isotropic vectors in V . Here we use the convention (3.23). For similar integrals of the Kudla-Millson theta series, also see [15] (signature $(p, 2)$) and [10] (signature (p, q)).

Now consider signature $(2, 1)$ and let $\eta = \eta_D$ be the differential of the third kind associated to the divisor $D := [\zeta_1] - [\zeta_2] \in \text{Div}^0(M^*)$. Here $\zeta_j \in M^*$ is the image of c_j for $j = 1, 2$ in M^* . Let $\gamma \subset M$ be the image of \mathcal{D}_{c_1,c_2} . When summing the signed intersection of c_X and \mathcal{D}_{c_1,c_2} over $X \in L_{m,h}$, we obtain a Γ -invariant quantity, which is equal to the signed intersection $\mathbf{I}(\gamma, [c(X)])$ (defined in §2.3) averaged over $X \in \Gamma \backslash L_{m,h}$.

It is clear from this definition and Theorem 3.7 that $-2\pi i \vartheta_L^{c_1,c_2}(\tau)$ and $I(\tau, \eta)$ have the same non-holomorphic part. The discrepancy in their holomorphic part can be explained geometrically using the Kudla-Millson lift of a closed 1-form. By Lemma 2.2, we can relate $\vartheta_L^{c_1,c_2}(\tau)$ to $I(\tau, \eta)$ as follows.

Proposition 4.1. *Fix a section $s : H_1(M^*, \mathbb{Z}) \rightarrow H_1(M^* \setminus \{\zeta_1, \zeta_2\}, \mathbb{Z})$ of the projection π and γ_s a path from ζ_1 to ζ_2 as in §2.3. Let $\omega_{\eta,s} \in H_{\text{DR}}^1(M^*)$ be the closed differential form satisfying (2.14). Then there exists a unary theta series $g_{\eta,s} \in S_{3/2,\rho_L}$ such that*

$$(4.3) \quad \vartheta_L^{c_1,c_2}(\tau) = -\frac{1}{2\pi i} (I(\tau, \eta) - I(\tau, \omega_{\eta,s}) + g_{\eta,s}(\tau)) + I(\tau, \omega_{[\gamma-\gamma_s]}),$$

where $\omega_{\mathbf{c}}$ denotes the Poincaré dual of $\mathbf{c} \in H_1(M^*, \mathbb{Z})$.

Proof. From the definition of $\vartheta_L^{c_1,c_2}$ and Theorem 3.7, we see that both sides have the same non-holomorphic part. For the holomorphic part, suppose $c(X)$ is closed for $X \in L_{m,h}$. This happens when $\sqrt{mN} \notin \mathbb{Q}^\times$ or M^* has only one cusp. We need to show

$$\sum_{X \in \Gamma_L \backslash L_{m,h}} \mathbf{I}(\gamma, [c(X)]) - \int_{c(X)} \omega_{[\gamma-\gamma_s]} = -\frac{1}{2\pi i} \sum_{X \in \Gamma_L \backslash L_{m,h}} \int_{c(X)} \eta - \omega_{\eta,s}$$

for all $h \in L'/L$. By definition of the signed intersection number, we know that $\int_{c(X)} \omega_{[\gamma-\gamma_s]} = \mathbf{I}(\gamma, [c(X)]) - \mathbf{I}(\gamma_s, [c(X)])$. Lemma 2.2 then finishes the proof. \square

Remark 4.2. If M^* has genus zero, resp. only one cusp, then $\omega_{\eta,s}$, resp. $g_{\eta,s}$, vanishes identically. Furthermore, if M^* is the modular curve as in the introduction, we obtain the equality of the twisted theta integral over η with the (twisted) ϑ^{c_1, c_2} .

5. SHIMURA CURVE AND MOCK THETA FUNCTION.

In this section, we will give an example where η is a differential of the third kind on a Shimura curve M and the generating series $I(\tau, \eta)$ is closely related to the third order mock theta function of Ramanujan.

Let B be a quaternion algebra over \mathbb{Q} with discriminant 6 [23, §3]. It is the \mathbb{Q} -algebra generated by α_0, α_1 satisfying

$$(5.1) \quad \alpha_0^2 = -1, \alpha_1^2 = 3, \alpha_2 := \alpha_0\alpha_1 = -\alpha_1\alpha_0.$$

As before, we let $Q = -\text{Nm}$ be the quadratic form. The subspace $B^0 \subset B$ of trace zero elements is given by $\mathbb{Q}\alpha_0 \oplus \mathbb{Q}\alpha_1 \oplus \mathbb{Q}\alpha_2$. It is more convenient to view B as a \mathbb{Q} -subalgebra of $M_2(\mathbb{R})$ via the embedding

$$(5.2) \quad \begin{aligned} \iota_\infty : B &\hookrightarrow M_2(\mathbb{R}) \\ \alpha_0, \alpha_1, \alpha_2 &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{3} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

This extends to an isometry between quadratic spaces $(B_\mathbb{R}, \text{Nm})$ and $(M_2(\mathbb{R}), \det)$. We slightly abuse the notation by using X to denote $\iota_\infty(X)$ for $X \in B_\mathbb{R}$.

We choose a maximal order $\mathcal{O} := \mathbb{Z} \oplus \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_3$ with $\alpha_3 := (1 + \alpha_0 + \alpha_1 + \alpha_2)/2$ and the units $\mathcal{O}_1^* \subset \mathcal{O}$ of reduced norm 1 act by conjugation on \mathcal{O} . We take as our lattice

$$(5.3) \quad L := B^0 \cap \mathcal{O} = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2.$$

Then $Q(\alpha_0) = -1, Q(\alpha_1) = Q(\alpha_2) = 3$ and

$$(5.4) \quad L' = \frac{1}{2}\mathbb{Z}\alpha_0 \oplus \frac{1}{6}\mathbb{Z}\alpha_1 \oplus \frac{1}{6}\mathbb{Z}\alpha_2, \quad L'/L = \bigoplus_{j=1}^3 L'_j/L_j, \quad L_j := \mathbb{Z}\alpha_j.$$

We identify L'_0/L_0 with $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ and $L'_1/L_1, L'_2/L_2$ with $\frac{1}{6}\mathbb{Z}/\mathbb{Z}$. The group $\mathcal{O}_1^* \subset \text{SO}(L)$ fixes the connected component of \mathcal{D} and can be viewed as a discrete subgroup of $\text{SL}_2(\mathbb{R})$ via the map ι_∞ above. It can be generated by $\alpha_0, \alpha_0 + \alpha_3, 2\alpha_0 + \alpha_2$, and naturally acts on L'/L with the image in $\text{Aut}(L'/L)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. Let $\Gamma := \mathcal{O}_1^* \cap \Gamma_L$ be the index 6 subgroup of \mathcal{O}_1^* . The quotient $M = \Gamma \backslash \mathbb{H}$ is a Shimura curve of genus one.

Let η be a differential of the third kind on M . For $m \in \mathbb{Q}$ and $h \in L'/L$, the quantity $\text{Tr}_{m,h}(\eta)$ is defined in (3.28). Theorem 3.7 then implies that the generating series

$$(5.5) \quad \sum_{h \in L'/L} \mathfrak{e}_h \sum_{m \in \mathbb{Q}} \text{Tr}_{m,h}(\eta) q^m$$

is the holomorphic part of a mixed mock modular form with shadow $\sum_{\zeta \in M} r_\zeta(\eta) \vartheta(\tau, \zeta)$, where $\text{res}(\eta) = \sum_{\zeta \in M} r_\zeta(\eta) [\zeta]$ and

$$\vartheta(\tau, z) := v \sum_{h \in L'/L} \mathbf{e}_h \vartheta_h(\tau, z), \quad \vartheta_h(\tau, z) := \sum_{X \in L+h} \frac{(X, X(z))}{2} \mathbf{e} \left(\frac{(X, X(z))^2}{4} \tau - \frac{|(X, X^\perp(z))|^2}{4} \bar{\tau} \right)$$

for $z = x + iy \in \mathbb{H}$, $X(z) = \frac{1}{y} \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}$ and $X^\perp(z) = \frac{1}{y} \begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix}$.

Let $z_0 = x_0 + iy_0 := \frac{1+\sqrt{-2}}{\sqrt{3}} \in \mathbb{H}$ be a CM point. Then

$$(5.6) \quad \begin{aligned} \mathbb{R}X(z_0) \cap L &= \mathbb{Z}\lambda_0, \quad (\mathbb{R}\text{Re}(X^\perp(z_0)) \oplus \mathbb{R}\text{Im}(X^\perp(z_0))) \cap L = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\alpha_2, \\ \lambda_0 &:= 3\alpha_0 + \alpha_1, \quad \lambda_1 := \alpha_0 + \alpha_1, \end{aligned}$$

and $\tilde{L} := \mathbb{Z}\lambda_0 \oplus \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\alpha_2$ is a sublattice of L of index 2. Denote the lattices $\mathbb{Z}\lambda_0, \mathbb{Z}\lambda_1$ by N, P respectively. Then $\tilde{L} = N \oplus P \oplus L_2$ and $\tilde{L}'/\tilde{L} = N'/N \oplus P'/P \oplus L_2'/L_2 \subset (\mathbb{Q}/\mathbb{Z})^3$. For $X = r_0\lambda_0 + r_1\lambda_1 + r_2\alpha_2 \in V$, it is easy to check that

$$(5.7) \quad (X, X(z_0)) = -\frac{2r_0}{\sqrt{6}}Q(\lambda_0) = 2\sqrt{6}r_0, \quad |(X, X^\perp(z_0))|^2 = 8r_1^2 + 12r_2^2.$$

Suppose $X \in \tilde{L}'$, then $X \in L'$ if and only if $3r_0 + r_1 \in \frac{1}{2}\mathbb{Z}$. This also implies $r_0 + r_1 \in \frac{1}{6}\mathbb{Z}$. Therefore, we have the following two-to-one surjective map

$$(5.8) \quad \begin{aligned} \phi : L'/\tilde{L} &\rightarrow L'/L \\ \mu = (\mu_0, \mu_1, \mu_2) &\mapsto (3\mu_0 + \mu_1, \mu_0 + \mu_1, \mu_2) \end{aligned}$$

where the addition is carried out in \mathbb{Q}/\mathbb{Z} . It is easy to check that if $X \in (\tilde{L} + \mu) \cap L' \subset V$, then $X \in L + \phi(\mu)$. This map induces a linear map $A_\phi : \mathbb{C}[L'/L] \rightarrow \mathbb{C}[\tilde{L}'/\tilde{L}]$ defined by

$$(5.9) \quad A_\phi(\mathbf{e}_h) := \sum_{\mu \in \phi^{-1}(h)} \mathbf{e}_\mu$$

for each $h \in L'/L$.

Define the theta functions

$$(5.10) \quad \begin{aligned} \vartheta_N(\tau) &:= \sum_{h \in N'/N} \mathbf{e}_h \sum_{\lambda \in N+h} \frac{(\lambda, \lambda_0)}{\sqrt{6}} \mathbf{e}(Q(\lambda)\tau), \quad \theta_P(\tau) := \sum_{h \in P'/P} \mathbf{e}_h \sum_{\lambda \in P+h} \mathbf{e}(Q(\lambda)\tau), \\ \theta_{L_2}(\tau) &:= \sum_{h \in L_2'/L_2} \mathbf{e}_h \sum_{\lambda \in L_2+h} \mathbf{e}(Q(\lambda)\tau). \end{aligned}$$

Notice that $\vartheta_{N, \mu_0}(\tau) = -\vartheta_{N, -\mu_0}(\tau)$ and $\theta_{P, \mu_1}(\tau) = \theta_{P, -\mu_1}(\tau)$ for $\mu_0 \in N'/N, \mu_1 \in P'/P$. For each $h = (h_0, h_1, h_2) \in L'/L$, the function $\vartheta_h(\tau, z_0)$ can be written as

$$\vartheta_h(\tau, z_0) = v \sum_{\mu = (\mu_0, \mu_1, \mu_2) \in L'/\tilde{L}, \phi(\mu) = h} \vartheta_{N, \mu_0}(\tau) \theta_{P, \mu_1}(-\bar{\tau}) \theta_{L_2, \mu_2}(-\bar{\tau}).$$

The same argument applies when z_0 is replaced by $-\overline{z_0}$. In that case, the equation above becomes

$$\vartheta_h(\tau, -\overline{z_0}) = v \sum_{\mu=(\mu_0, \mu_1, \mu_2) \in L'/\tilde{L}, \phi(\mu)=h} \vartheta_{N, \sigma(\mu_0)}(\tau) \theta_{P, \mu_1}(-\overline{\tau}) \theta_{L_2, \mu_2}(-\overline{\tau}),$$

where $\sigma \in \text{Iso}(N'/N)$ sends $\mu_0 \in N'/N = \frac{1}{12}\mathbb{Z}$ to $5\mu_0 \in N'/N$. Since σ is an isometry, it defines an automorphism on $\mathcal{A}_k(\rho_N)$, which we also use σ to denote.

Let η a differential of the third kind on M with the residue divisor

$$(5.11) \quad \text{res}(\eta) = [z_0] - [-\overline{z_0}].$$

Then the generating series (5.5) is the holomorphic part of an automorphic form in $\mathcal{A}_{3/2}(\rho_L)$, whose image under $\xi_{3/2}$ is

$$(5.12) \quad vA_\phi^*((\vartheta_{N, \sigma}(\tau)) \otimes \theta_P(-\overline{\tau}) \otimes \theta_P(-\overline{\tau})) \in \mathcal{A}_{1/2}(\overline{\rho_L}),$$

where $\vartheta_{N, \sigma}(\tau) := \vartheta_N(\tau) - \sigma(\vartheta_N(\tau))$.

On the other hand, recall the following third order mock theta functions of Ramanujan

$$(5.13) \quad f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^n (1+q^k)^2}, \quad \omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{\prod_{k=1}^n (1-q^{2k-1})^2}.$$

After setting $q = \mathbf{e}(\tau)$ and defining

$$(5.14) \quad f_0(\tau) := q^{-1/24} f(q), \quad f_1(\tau) := 2q^{1/3} \omega(q^{1/2}), \quad f_2 := 2q^{1/3} \omega(-q^{1/2}),$$

Zwegers showed that the column vector $(f_0, f_1, f_2)^T$ is a mock modular form with a unary theta function as its shadow [28]. One can use this to show that the vector (see [6])

$$\tilde{\vartheta}_{N, \sigma}^+ := -\frac{1}{4}(0, f_0, f_2 - f_1, 0, -f_1 - f_2, -f_0, 0, f_0, f_1 + f_2, 0, f_1 - f_2, -f_0)^T$$

is the holomorphic part of a harmonic Maass form $\tilde{\vartheta}_{N, \sigma}$, whose image under $\xi_{3/2}$ is $\vartheta_{N, \sigma}$. This has a pole of the form $q^{-1/24} + O(1)$ at the components \mathfrak{e}_{μ_0} for $\mu_0 \in \{\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}\} \subset N'/N$. In these cases, $\mu_1 \in \{\frac{1}{4}, \frac{3}{4}\} \subset P'/P$ in order for $(\mu_0, \mu_1, \mu_2) \in L'/\tilde{L}$. For these μ_1 , the function $\theta_{P, \mu_1}(\tau)$ has the form $q^{1/8} + O(q)$. Therefore for any $h \in L'/L$, the function

$$\sum_{\mu=(\mu_0, \mu_1, \mu_2) \in L'/\tilde{L}, \phi(\mu)=h} \tilde{\vartheta}_{N, \sigma, \mu_0}(\tau) \theta_{P, \mu_1}(\tau) \theta_{L_2, \mu_2}(\tau)$$

decays exponentially at the cusp. Therefore, there exists a cusp form $g \in S_{3/2}(\rho_L)$ such that

$$\frac{I(\tau, \eta)}{2\pi i} = A_\phi^*(\tilde{\vartheta}_{N, \sigma}(\tau) \otimes \theta_P(\tau) \otimes \theta_P(\tau)) + g(\tau).$$

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